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The Ricci tensor of SU(3)-manifolds

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Abstract

Following the approach of Bryant [R. Bryant, Some remarks on G_2-structures. e-print: math.DG/0305124] we study the intrinsic torsion of a SU(3)-manifold deriving a number of formulae for the Ricci and the scalar curvature in terms of torsion forms. As a consequence we prove that in some special cases the Einstein condition forces the vanishing of the intrinsic torsion.

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0. Introduction

In the last few years geometric and physical motivations led many mathematicians to focus on the geometry of SU(3)- and G_2-structures on 6- and 7-dimensional manifolds and on the interplay between them (see e.g. [2–5,10–14,20] and the references therein). New directions in this field were suggested by the work of Hitchin [22]. The present work is inspired by [10], where the author computes the Ricci curvature of a G_2-structure in terms of the derivatives of the defining 3-form.

In this paper we study the intrinsic torsion of SU(3)-manifolds relating it to the curvature of the induced metric. A SU(3)-structure on a 6-dimensional manifold is determined by a pair (κ, Ω), where κ is an almost symplectic structure and Ω is a normalized κ-positive 3-form (see Section 2 for the definition). In fact such a pair induces a natural κ-calibrated almost complex structure J on M such that the complex valued form

$$\varepsilon = \Omega + \mathrm{i}\, J\Omega$$

is of type (3,0) with respect to J. The intrinsic torsion of a SU(3)-structure can be described in terms of the derivatives of the defining forms (κ, Ω) by considering a natural decomposition of $\Lambda^3 M$ and $\Lambda^4 M$ in irreducible SU(3)-submodules. Namely the forms $\mathrm{d}\kappa$, $\mathrm{d}\Omega$ and $\mathrm{d}^*\Omega$ decompose as

$$\mathrm{d}\kappa = -\frac{3}{2}\sigma_0\Omega + \frac{3}{2}\pi_0 J\Omega + \nu_1\wedge\kappa + \nu_3;$$

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$$d\Omega = \pi_0\kappa^2 + \pi_1 \wedge \Omega - \pi_2 \wedge \kappa;$$
$$dJ\Omega = \sigma_0\kappa^2 + J\pi_1 \wedge \Omega - \sigma_2 \wedge \kappa,$$

where $\pi_0, \sigma_0, \pi_1, \nu_1, \sigma_2, \nu_3$ lie in different SU(3)-modules. The forms $\{\pi_0, \sigma_0, \pi_1, \nu_1, \sigma_2, \nu_3\}$ are called the *torsion forms* and they vanish if and only if the SU(3)-structure is integrable, i.e. if and only if the induced metric is Ricci-flat so that (M, κ, Ω) is a Calabi–Yau threefold. Moreover special non-integrable SU(3)-structures, e.g. generalized Calabi–Yau structures[1] and half-flat structures, can be characterized in terms of torsion forms. In the spirit of [10] a principal bundle approach allows us to write down the Ricci tensor and the scalar curvature of a SU(3)-manifold in terms of torsion forms. As a direct consequence of these formulae we get that the scalar curvature of a generalized Calabi–Yau manifold is non-positive and it vanishes identically if and only if the SU(3)-structure is integrable. We also prove that the metric of a special generalized Calabi–Yau manifold M is Einstein if and only if M is a genuine Calabi–Yau manifold.

The paper is organized as follows. In Section 1 general SU(n)-structures are introduced. In Section 2, which is the algebraic core of the paper, we specialize to the 6-dimensional case studying the algebra underlying SU(3)-structures. In particular we exhibit an explicit expression for the complex structure induced by (κ, Ω). In this section we define the torsion forms and characterize various special SU(3)-structures in terms of these forms. The work in Section 3 follows the steps of [10] where the formula for the Ricci curvature of a G_2-structure is derived. We exploit the algebraic formulae obtained in Section 2 in order to come to the explicit formula for the Ricci tensor (3.13). Here the final computation was carried out with the aid of MAPLE while a representation-theoretic argument justifies the final formulae. In Section 4 we collect the above mentioned consequences of formula (3.13) in the special case of generalized Calabi–Yau manifolds. Section 5 is devoted to the explicit computations performed on a non-integrable special generalized Calabi–Yau nilmanifold which illustrate the role of the torsion forms in this case. In the Appendix some technical proofs are provided.

NOTATION. Given a manifold M, we denote by $\Lambda^r M$ the space of smooth r-forms on M and we set $\Lambda^\bullet M := \bigoplus_{r=1}^n \Lambda^r M$. When an almost complex structure J on M is given, $\Lambda_J^{p,q} M$ denotes the space of complex forms on M of type (p, q) with respect to J.

The symplectic group, i.e. the group of automorphisms of $\mathbb{R}^{2n}$ preserving the standard symplectic form $\kappa_n = \sum_{i=1}^n dx_{2i-1} \wedge dx_{2i}$, will be denoted by $\mathrm{Sp}(n, \mathbb{R})$.

Furthermore when a coframe $\{\alpha_1, \dots, \alpha_n\}$ is given we will denote the r-form $\alpha_{i_1} \wedge \cdots \wedge \alpha_{i_r}$ by $\alpha_{i_1 \dots i_r}$.

In the indicial expressions the symbol for sum over repeated indices is omitted.

1. SU(n)-structures

1.1. U(n)-structures

Let (M, κ) be a $2n$-dimensional almost symplectic manifold. The *symplectic Hodge operator*

$$\bigstar : \Lambda^r M \to \Lambda^{2n-r} M,$$

is defined by means of the relation

$$\alpha \wedge \bigstar\beta = \kappa(\alpha, \beta)\frac{\kappa^n}{n!},$$

where $\alpha, \beta \in \Lambda^r M$. It is easy to check that $\bigstar^2 = I$. An almost complex structure on M is an endomorphism J of TM such that $J^2 = -I$. Note that the endomorphism induced by J on $\Lambda^p M$ (again denoted by J) satisfies the identity $J^2 = (-1)^p I$. An almost complex structure is said to be κ-*tamed* if

$$\kappa_x(v, J_x v) > 0$$

[1] We remark that the notion of generalized Calabi–Yau structure that we consider is the one adopted in [18] which is different from that given by Hitchin in [21].

for every $x \in M$ and non-zero vector $v \in T_xM$. If further κ is preserved by J, the almost complex structure is said to be κ*-calibrated*. In this case we denote by g_J the Riemannian metric

$$g_J(X, Y) := \kappa(X, JY), \tag{1.1}$$

for every vector field X, Y on M. We immediately get that J is an isometry of g_J, i.e. g_J is J-Hermitian. We denote by $\mathcal{C}_\kappa(M)$ the space of κ-calibrated almost complex structures on M. The elements of $\mathcal{C}_\kappa(M)$ can be viewed as smooth global sections of a fiber bundle whose fibers are isomorphic to the homogeneous space

$$\mathrm{Sp}(n, \mathbb{R})/\mathrm{U}(n)$$

(see e.g. [6]). Since the latter is topologically a $(n + n^2)$-dimensional cell, given any almost symplectic form κ, there are always plenty of κ-calibrated almost complex structures. Furthermore the fact that $\mathcal{C}_\kappa(M)$ is contractible makes it possible to define the first Chern class $c_1(M, \kappa)$ of the almost symplectic manifold (M, κ) as $c_1(M, J)$, where $J \in \mathcal{C}_\kappa(M)$.

Given $J \in \mathcal{C}_\kappa(M)$ the complexified exterior algebra $\Lambda^\bullet M \otimes \mathbb{C}$ is $\mathbb{Z}^+$-bigraded with respect to the type as

$$\Lambda^\bullet M \otimes \mathbb{C} = \bigoplus_{r=0}^{2n} \bigoplus_{p+q=r} \Lambda_J^{p,q} M.$$

The metric g_J together with the orientation given by κ defines also the classical *Hodge operator*, that in this setting is a $\mathbb{C}$-linear map $* : \Lambda_J^{p,q} M \to \Lambda_J^{n-q,n-p} M$, such that

$$\alpha \wedge \overline{*\beta} = g_J(\alpha, \overline{\beta}) \frac{\kappa^n}{n!},$$

for all $\alpha, \beta \in \Lambda_J^{p,q} M$. It is well known that $*$ commutes with J and that their composition equals the $\mathbb{C}$-linear extension of the symplectic Hodge operator:

$$*J = J* = \bigstar.$$

Since we have

$$\mathrm{d} : \Lambda_J^{p,q} M \to \Lambda_J^{p+2,q-1} M \oplus \Lambda_J^{p+1,q} M \oplus \Lambda_J^{p,q+1} M \oplus \Lambda_J^{p-1,q+2} M,$$

the exterior differential operator accordingly splits as

$$\mathrm{d} = A_J + \partial_J + \bar{\partial}_J + \bar{A}_J.$$

It is well known that an almost complex structure is integrable if and only if $\bar{A}_J = 0$.

1.2. SU(n)-structures

Let M be a $2n$-dimensional manifold and $\mathcal{L}(M)$ be the $\mathrm{GL}(2n, \mathbb{R})$-principal bundle of linear frames. A $\mathrm{SU}(n)$-structure on M is a $\mathrm{SU}(n)$-reduction of $\mathcal{L}(M)$. Since $\mathrm{SU}(n)$ is the group of the unitary transformation of $\mathbb{C}^n$ preserving the standard complex volume form, a $\mathrm{SU}(n)$-structure on M is determined by the choice of the following data:

- an almost complex structure J on TM;
- a J-Hermitian metric g;
- a complex $(n, 0)$-form ε of constant norm $2^{\frac{n}{2}}$.

Alternatively these data can be replaced by

- an almost symplectic structure κ;
- a κ-calibrated almost complex structure J;
- a complex $(n, 0)$-form ε, satisfying $\varepsilon \wedge \overline{\varepsilon} = c_n \frac{\kappa^n}{n!}$, with $c_n = (-1)^{\frac{n(n+1)}{2}} (2\mathrm{i})^n$;

where κ and g are related by (1.1). Denote by ∇ the Levi-Civita connection induced by g on TM. We will say that a SU(n)-structure is *integrable* if the restricted holonomy group $\mathrm{Hol}^0(TM, \nabla)$ is isomorphic to a subgroup of SU(n). Since the holonomy is determined by the parallel tensors, a SU(n)-structure is integrable if the corresponding triple (κ, J, ε) satisfies

$$\nabla\kappa = 0, \qquad \nabla J = 0, \qquad \nabla\varepsilon = 0.$$

In this case $(M, \kappa, J, \varepsilon)$ is said to be a *Calabi–Yau manifold*.

Remark 1.1. Let $(M, \kappa, J, \varepsilon)$ be a SU(n)-manifold and assume

$$\mathrm{d}\kappa = 0, \qquad \mathrm{d}\varepsilon = 0,$$

then $(M, \kappa, J, \varepsilon)$ is a Calabi–Yau manifold. In fact if $\alpha \in \Lambda_J^{1,0}M$ we have

$$0 = \mathrm{d}(\varepsilon \wedge \alpha) = (-1)^n \varepsilon \wedge \mathrm{d}\alpha = (-1)^n \varepsilon \wedge \overline{A}_J \alpha,$$

and hence $\overline{A}_J = 0$, which implies that J is integrable. Furthermore, since κ is closed, the pair (κ, J) defines a Kähler structure on M; hence we get

$$\nabla\kappa = 0, \qquad \nabla J = 0.$$

Finally the equation $\varepsilon \wedge \overline{\varepsilon} = c_n \frac{\kappa^n}{n!}$ forces ε to be parallel.

Several non-integrable SU(n)-structures are worth considering for both geometrical and physical reasons (the survey article [1] is a good reference for recent results on non-integrable geometries).
A notion of generalized Calabi–Yau manifold has been introduced by de Bartolomeis and Tomassini; in [18] they give the following definition:

Definition 1.2. A *generalized Calabi–Yau (GCY) structure* on M is a SU(n)-structure (κ, J, ε) satisfying the following conditions:

1. $\mathrm{d}\kappa = 0$ (i.e. (M, κ) is a symplectic manifold);
2. $\overline{\partial}_J \varepsilon = 0$.

We emphasize again that a different generalization of Calabi–Yau structures has been considered by Hitchin in a broader context in [21].

Remark 1.3. For an almost Kähler manifold (i.e. a symplectic manifold endowed with a calibrated almost complex structure) it is natural to consider on TM the canonical Hermitian connection $\widetilde{\nabla}$, whose covariant derivative is given by

$$\widetilde{\nabla}_X = \nabla_X - \frac{1}{2} J \nabla_X J.$$

It is characterized by the following properties

$$\widetilde{\nabla}\kappa = 0, \qquad \widetilde{\nabla} J = 0, \qquad T^{\widetilde{\nabla}} = \frac{1}{2} N_J,$$

where N_J is the Nijenhuis tensor associated with J and $T^{\widetilde{\nabla}}$ is the torsion of $\widetilde{\nabla}$. This connection coincides with ∇ if and only if the pair (κ, J) is a Kähler structure on M (i.e. if and only if J is integrable).

If $(M, \kappa, J, \varepsilon)$ is a symplectic SU(3)-manifold, then the constraint $\varepsilon \wedge \overline{\varepsilon} = c_n \frac{\kappa^n}{n!}$ implies

$$\overline{\partial}_J \varepsilon = 0 \iff \widetilde{\nabla}\varepsilon = 0,$$

(see [18]). Hence GCY manifolds can be defined as SU(n)-manifolds with the volume form ε satisfying $\widetilde{\nabla}\varepsilon = 0$. It follows that in the GCY case the holonomy group $\mathrm{Hol}^0(TM, \widetilde{\nabla})$ is isomorphic to a subgroup of SU(n).

2. SU(3)-structures

In this section we specialize to the case $n = 3$ and study the linear algebra underlying SU(3)-structures. Fix a real 6-dimensional symplectic vector space (V, κ). Let us denote by $\mathrm{Sp}(V, \kappa)$ the group of automorphisms of the pair (V, κ), i.e. $\mathrm{Sp}(V, \kappa) = \{\phi \in \mathrm{GL}(V) : \phi^*\kappa = \kappa\}$. The space of skew-symmetric 3-forms on V splits into the following two irreducible $\mathrm{Sp}(V, \kappa)$-modules

$$\Lambda^3_0 V^* = \{\phi \in \Lambda^3 V^* \mid \phi \wedge \kappa = 0\},$$
$$\Lambda^3_6 V^* = \{\alpha \wedge \kappa \mid \alpha \in V^*\}.$$

The 3-forms lying in the space $\Lambda^3_0 V^*$ are sometimes called in the literature *effective* 3-forms (see e.g. [7]). Let us consider the action Θ of the Lie group $G = \mathrm{Sp}(V, \kappa) \times \mathbb{R}^*_+$ on the space $\Lambda^3_0 V^*$ given by

$$\Theta(\phi, t) \cdot \alpha := t(\phi^{-1})^*\alpha,$$

where $\mathbb{R}^*_+$ denotes the group of positive real numbers. It is known that this action has an open orbit $\mathcal{O}$ whose isotropy is locally isomorphic to SU(3) (see e.g. [7,24]). We will call *κ-positive* 3-forms the elements of the orbit $\mathcal{O}$. Since the stabilizer at $\Omega \in \mathcal{O}$ is locally isomorphic to SU(3), each κ-positive 3-form singles out a κ-calibrated complex structure on V which we are able to explicitly write down. In fact we have:

Proposition 2.1. *The endomorphism P_Ω of V^* given by*

$$P_\Omega : \alpha \longmapsto -\frac{1}{2}\bigstar(\Omega \wedge \bigstar(\Omega \wedge \alpha))$$

has the following properties

1. *P^2_Ω is a negative multiple of the identity;*
2. *$\kappa(P_\Omega \alpha, \beta) = -\kappa(\alpha, P_\Omega \beta)$, for every $\alpha, \beta \in \Lambda^1 V^*$.*

Proof. 1. First we observe that P_Ω is a SU(3)-invariant endomorphism of V^*, since it is built using only Ω and $\bigstar$. Since SU(3) acts irreducibly on V^*, the real version of Schur's lemma assures that $P_\Omega = aI + bJ$, where J is a complex structure on V^* and a, b are real numbers.
Now we claim that P^2_Ω has a negative eigenvalue. From this claim the conclusion follows. Suppose indeed that there exists $v \neq 0$ such that $P^2_\Omega v = \lambda v$, with $\lambda < 0$. Then

$$2abJv = (\lambda^2 - a^2 + b^2)v.$$

If $ab \neq 0$, then J would have a real eigenvalue and this is impossible. On the other hand if $b = 0$ then $P^2_\Omega = a^2 I$, which is a contradiction with the claim. Hence $P_\Omega = bJ$. To prove the claim we must use an explicit frame $\{e^1, \ldots, e^6\}$ of V^* in which κ and Ω takes the standard form and perform the computation, e.g., of $P^2_\Omega e^1$.

2. We have

$$\begin{aligned}\kappa(P_\Omega\alpha, \beta)\frac{\kappa^3}{6} &= -\kappa(\beta, P_\Omega\alpha)\frac{\kappa^3}{6} = \frac{1}{2}\beta \wedge \Omega \wedge \bigstar(\Omega \wedge \alpha)\\ &= -\frac{1}{2}\kappa(\beta \wedge \Omega, \alpha \wedge \Omega)\frac{\kappa^3}{6} = -\frac{1}{2}\kappa(\alpha \wedge \Omega, \beta \wedge \Omega)\frac{\kappa^3}{6}\\ &= \kappa(P_\Omega\beta, \alpha)\frac{\kappa^3}{6} = -\kappa(\alpha, P_\Omega\beta)\frac{\kappa^3}{6}. \quad \square\end{aligned}$$

The following is immediate:

Corollary 2.2. *The endomorphism J_Ω κ-dual to* $(\det P_\Omega)^{-\frac{1}{6}} P_\Omega$ *is a κ-calibrated almost complex structure on V. Furthermore the form*

$$\varepsilon = \Omega + \mathrm{i}J_\Omega\Omega$$

is a complex form of type $(3,0)$ *with respect to* J_Ω. *If further* $\det(P_\Omega)=1$, *then*

$$\varepsilon\wedge\overline{\varepsilon}=\mathrm{i}\frac{4}{3}\kappa^3. \tag{2.1}$$

We have also this characterization of κ-positive 3-forms.

Lemma 2.3. *These facts are equivalent*

1. Ω *is a* κ*-positive 3-form;*
2. *the map* $F_\Omega:\Lambda^1V^*\ni\alpha\mapsto\alpha\wedge\Omega$ *is injective and* κ *is negative definite on the image of* F_Ω.

Remark 2.4. Note that since κ is J_Ω-invariant, also $J_\Omega\Omega$ is effective, i.e. $\kappa\wedge J_\Omega\Omega=0$.

Definition 2.5. A κ-positive 3-form is said to be *normalized* if $\det(P_\Omega)=1$.

From now on we will drop the subscript Ω from J_Ω when no confusion arises.
In order to make the exposition more concrete we identify V with $\mathbb{R}^6$; we denote by $\{e_1,\dots,e_6\}$ the standard basis and by $\{e^1,\dots,e^6\}$ the dual one.
Fix on V the standard symplectic form

$$\kappa_0=e^{12}+e^{34}+e^{56}$$

and the standard complex volume form

$$\varepsilon_0=(e^1+\mathrm{i}e^2)\wedge(e^3+\mathrm{i}e^4)\wedge(e^5+\mathrm{i}e^6).$$

The real part of ε_0

$$\Omega_0=e^{135}-e^{146}-e^{245}-e^{236}$$

is a normalized κ_0-positive 3-form. The complex structure associated with Ω_0 is exactly the standard κ_0-calibrated complex structure J_0 defined by

$$J_0(e_1)=e_2,\qquad J_0(e_3)=e_4,\qquad J_0(e_5)=e_6.$$

We will denote by g_0 the scalar product associated with (κ_0,J_0). Note that g_0 is simply the standard Euclidean inner product.

Using the standard forms κ_0 and Ω_0 by straightforward computations we can obtain some useful identities concerning κ-positive 3-forms.

Lemma 2.6. *Let* (V,κ) *be a symplectic vector space and* Ω *a normalized* κ*-positive* 3*-form; then we have*

1. $\bigstar\Omega=-\Omega$ *(hence also* $J\Omega=*\Omega$*);*
2. $\Omega\wedge J\Omega=\frac{2}{3}\kappa^3$.

2.1. Decomposition of the exterior algebra

Let (V,κ) be an arbitrary 6-dimensional symplectic vector space and Ω a normalized κ-positive 3-form. Let us consider the natural action of SU(3) on the exterior algebra $\Lambda^\bullet V^*$. Obviously SU(3) acts irreducibly on V^* and Λ^5V^*, while Λ^2V^* and Λ^3V^* decompose as follows:

$$\begin{aligned}\Lambda^2V^*&=\Lambda^2_1V^*\oplus\Lambda^2_6V^*\oplus\Lambda^2_8V^*,\\ \Lambda^3V^*&=\Lambda^3_{\mathrm{Re}}V^*\oplus\Lambda^3_{\mathrm{Im}}V^*\oplus\Lambda^3_6V^*\oplus\Lambda^3_{12}V^*,\end{aligned} \tag{2.2}$$

where we set

- $\Lambda^2_1V^*=\mathbb{R}\kappa$,
- $\Lambda^2_6V^*=\{\bigstar(\alpha\wedge\Omega)\mid\alpha\in\Lambda^1V^*\}=\{\varphi\in\Lambda^2V^*\mid J\varphi=-\varphi\}$,
- $\Lambda^2_8V^*=\{\varphi\in\Lambda^2V^*\mid\varphi\wedge\Omega=0\text{ and }\bigstar\varphi=-\varphi\wedge\kappa\}=\{\varphi\in\Lambda^2V^*\mid J\varphi=\varphi,\varphi\wedge\kappa^2=0\}$,

and

- $\Lambda^3_{\mathrm{Re}} V^* = \mathbb{R}\Omega$,
- $\Lambda^3_{\mathrm{Im}} V^* = \mathbb{R}J\Omega = \{\gamma \in \Lambda^3 V^* \mid \gamma \wedge \kappa = 0, \gamma \wedge \Omega = c\kappa^3, c \in \mathbb{R}\}$,
- $\Lambda^3_6 V^* = \{\alpha \wedge \kappa \mid \alpha \in \Lambda^1 V^*\} = \{\gamma \in \Lambda^3 V^* \mid \bigstar\gamma = \gamma\}$,
- $\Lambda^3_{12} V^* = \{\gamma \in \Lambda^3 V^* \mid \gamma \wedge \kappa = 0, \gamma \wedge \Omega = 0, \gamma \wedge J\Omega = 0\}$.

Remark 2.7. Now we emphasize some relations which will be useful:

1. If $\varphi \in \Lambda^2_6 V^* \oplus \Lambda^2_8 V^*$, then $\bigstar\varphi = -\varphi \wedge \kappa$.
2. If $\gamma \in \Lambda^3_{\mathrm{Re}} V^* \oplus \Lambda^3_{\mathrm{Im}} V^* \oplus \Lambda^3_{12} V^*$, then $\bigstar\gamma = -\gamma$ and $\gamma \wedge \kappa = 0$.
3. If α is an arbitrary 1-form, then $J(\alpha \wedge \Omega) = -\alpha \wedge \Omega$, and consequently from the definition of J it follows that
$$J\Omega \wedge \bigstar(\Omega \wedge \alpha) = -2\bigstar\alpha.$$
4. If $\beta \in \Lambda^2_8 V^*$ then
$$\begin{aligned} *(\beta \wedge \beta) \wedge \kappa^2 &= \beta \wedge \beta \wedge *\kappa^2 = 2\beta \wedge \beta \wedge \kappa \\ &= -2\beta \wedge \bigstar\beta = -2|\beta|^2 \frac{\kappa^3}{6}, \end{aligned}$$
so that
$$*(\kappa^2 \wedge *(\beta \wedge \beta)) = -2|\beta|^2. \tag{2.3}$$

We can obtain the decomposition of $\Lambda^4 V^*$ using the duality given by the symplectic star operator.

Moreover we define the projections

$$E_1 : \Lambda^2 V^* \to \Lambda^2_8 V^*,$$
$$E_2 : \Lambda^3 V^* \to \Lambda^3_{12} V^*$$

by

$$E_1(\alpha) = \frac{1}{2}(\alpha + J\alpha) - \frac{1}{18} * ((*(\alpha + J\alpha) + (\alpha + J\alpha) \wedge \kappa) \wedge \kappa)\kappa, \tag{2.4}$$

$$E_2(\beta) = \beta - \frac{1}{2} * (J\beta \wedge \kappa) \wedge \kappa - \frac{1}{4} * (\beta \wedge J\Omega)\Omega - \frac{1}{4} * (\Omega \wedge \beta)J\Omega. \tag{2.5}$$

Note that E_2 commutes with $*$ since the latter is an automorphism of $\Lambda^3_{12} V^*$. The same is true for J (and hence also for $\bigstar$).

2.2. The ϵ-identities

As done by Bryant in the G_2 case we introduce the following ϵ-notation, which will be useful in the sequel.

$$\Omega_0 = \frac{1}{6}\epsilon_{ijk}\, e^{ijk}, \qquad *\Omega_0 = \frac{1}{6}\overline{\epsilon}_{ijk}\, e^{ijk}, \qquad \kappa_0 = \frac{1}{2}\kappa_{ij}\, e^{ij}.$$

We will use the following identities, whose proof is straightforward:

$$\begin{aligned} &\epsilon_{ipq}\kappa_{pq} = 0; \\ &\kappa_{ip}\kappa_{pj} = -\delta_{ij}; \\ &\epsilon_{ijp}\kappa_{pr} = \overline{\epsilon}_{ijr}; \\ &\overline{\epsilon}_{ijp}\kappa_{pr} = -\epsilon_{ijr}; \\ &\overline{\epsilon}_{ipq}\epsilon_{jpq} = -4\kappa_{ij}; \\ &\epsilon_{ipq}\epsilon_{jpq} = 4\delta_{ij} = \overline{\epsilon}_{ipq}\overline{\epsilon}_{jpq}; \\ &\overline{\epsilon}_{ijp}\epsilon_{klp} = -\kappa_{ik}\delta_{jl} + \kappa_{jk}\delta_{il} + \kappa_{il}\delta_{jk} - \kappa_{jl}\delta_{ik}; \\ &\epsilon_{ijp}\epsilon_{klp} = -\kappa_{ik}\kappa_{jl} + \kappa_{il}\kappa_{jk} + \delta_{ik}\delta_{jl} - \delta_{jk}\delta_{il} = \overline{\epsilon}_{ijk}\overline{\epsilon}_{ipq}. \end{aligned} \tag{2.6}$$

These equations will be called *ϵ-identities*. As a first application of these formulae we can decompose the Lie algebra $\mathfrak{so}(6)$ as follows. Consider the real representation of complex matrices induced by J_0

$$\rho : \mathfrak{gl}(3, \mathbb{C}) \to \mathfrak{gl}(6, \mathbb{R}),$$

where $\rho(A)$ is the block matrix $(B_{ij})_{i,j=1,2,3}$, with $B_{ij} = \begin{pmatrix} \mathrm{Re}\, a_{ij} & \mathrm{Im}\, a_{ij} \\ -\mathrm{Im}\, a_{ij} & \mathrm{Re}\, a_{ij} \end{pmatrix}$. Thus a matrix $A = (a_{ij})$ lies in $\mathfrak{su}(3)$ if and only if

$$\epsilon_{ijk} a_{jk} = 0 \quad \text{and} \quad \kappa_{jk} a_{jk} = 0.$$

So we have the decomposition

$$\mathfrak{so}(6) = \mathfrak{su}(3) \oplus [\mathbb{R}]_1 \oplus [\mathbb{R}^6]_2,$$

where

$$([a]_1)_{ij} = a\kappa_{ij}, \qquad ([v]_2)_{ij} = \epsilon_{ijp} v_p.$$

2.3. Decomposition of symmetric 2-tensors

In order to express the Ricci tensor in terms of skew-symmetric forms we must establish the correspondence which we are going to describe. The 21-dimensional space of the symmetric covariant 2-tensor on V splits into irreducible $\mathfrak{su}(3)$-modules as follows:

$$S^2 V^* = \mathbb{R} g_0 \oplus S^2_+ \oplus S^2_-,$$

where

$$S^2_+ = \{h \in S^2 V^* : J_0 h = h, \mathrm{tr}_{g_0} h = 0\},$$
$$S^2_- = \{h \in S^2 V^* : J_0 h = -h\}.$$

We will denote by S^2_0 the direct sum $S^2_+ \oplus S^2_-$.

The maps

$$\iota : S^2_+ \longrightarrow \Lambda^2_8 V^*,$$
$$\gamma : S^2_- \longrightarrow \Lambda^3_{12} V^*$$

defined by

$$\iota(h_{ij} e^i e^j) = h_{ip} \kappa_{pj} e^{ij},$$
$$\gamma(h_{ij} e^i e^j) = h_{ip} \epsilon_{pjk} e^{ijk}$$

are isomorphisms of $\mathfrak{su}(3)$-representations.

2.4. SU(3)-structures on manifolds

Let M be a 6-dimensional manifold. A SU(3)-structure on M is determined by the choice of:

- a non-degenerate 2-form κ,
- a normalized κ-positive 3-form Ω (i.e. $\Omega[x]$ is $\kappa[x]$-positive and normalized at every x in M).

In fact, as we have seen, Ω determines a κ-calibrated almost complex structure J such that $\varepsilon = \Omega + \mathrm{i} J\Omega$ is of type $(3, 0)$ and satisfies Eq. (2.1). We refer to ε as to the *complex volume of* (κ, Ω). In the sequel the induced scalar product will be denoted by g or alternatively by $\langle , \rangle$ and the associated Hodge operator by $*$.
Note that the SU(3)-structure determined by (κ, Ω) is integrable if and only if

$$\mathrm{d}\kappa = 0, \qquad \mathrm{d}\Omega = \mathrm{d}^*\Omega = 0. \tag{2.7}$$

In fact, since $J\Omega = *\Omega$, Eq. (2.7) are equivalent to

$$d\kappa = 0, \qquad d\varepsilon = 0.$$

Hence, since $\varepsilon \wedge \overline{\varepsilon} = i\frac{4}{3}\kappa^3$, Remark 1.1 implies

$$\nabla\kappa = 0, \qquad \nabla J = 0, \qquad \nabla\varepsilon = 0 \iff d\kappa = 0, \qquad d\varepsilon = 0.$$

2.5. Torsion forms

Let (M, κ, Ω) be a SU(3)-manifold. According with (2.2) the space of r-forms splits into $\mathfrak{su}(3)$-modules as follows:

$$\Lambda^2 M = \Lambda^2_1 M \oplus \Lambda^2_6 M \oplus \Lambda^2_8 M,$$
$$\Lambda^3 M = \Lambda^3_{\mathrm{Re}} M \oplus \Lambda^3_{\mathrm{Im}} M \oplus \Lambda^3_6 M \oplus \Lambda^3_{12} M,$$
$$\Lambda^4 M = \Lambda^4_1 M \oplus \Lambda^4_6 M \oplus \Lambda^4_8 M,$$

where the meaning of the symbols is obvious. Consequently the derivatives of the structure forms decompose as

$$\begin{aligned} d\kappa &= \nu_0\Omega + \alpha_0 J\Omega + \nu_1 \wedge \kappa + \nu_3, \\ d\Omega &= \pi_0\kappa^2 + \pi_1 \wedge \Omega - \pi_2 \wedge \kappa, \\ dJ\Omega &= \sigma_0\kappa^2 + \sigma_1 \wedge \Omega - \sigma_2 \wedge \kappa, \end{aligned} \tag{2.8}$$

where $\nu_0, \alpha_0, \pi_0, \sigma_0 \in C^\infty(M, \mathbb{R})$, $\nu_1, \pi_1, \sigma_1 \in \Lambda^1 M$, $\pi_2, \sigma_2 \in \Lambda^2_8 M$ and $\nu_3 \in \Lambda^3_{12} M$.
The following equations are derived from a G_2 formula which was obtained in [9].

Lemma 2.8. *With the notation introduced above*

$$J\Omega \wedge (*dJ\Omega) - (*d\Omega) \wedge \Omega = 0. \tag{2.9}$$

Proof. See the Appendix. □

Now we are able to prove the following

Theorem 2.9. *The following relations hold:*

1. $\pi_0 = \frac{2}{3}\alpha_0$,
2. $\sigma_0 = -\frac{2}{3}\nu_0$,
3. $\sigma_1 = J\pi_1$.

Proof. 1. From the relation $\Omega \wedge \kappa = 0$ it follows that

$$\begin{aligned} 0 &= d(\Omega \wedge \kappa) = d\Omega \wedge \kappa - \Omega \wedge d\kappa \\ &= \pi_0\kappa^3 - \pi_2 \wedge \kappa^2 - \alpha_0\Omega \wedge J\Omega - \Omega \wedge \nu_3 \\ &= \left(\pi_0 - \frac{2}{3}\alpha_0\right)\kappa^3, \end{aligned}$$

where we have used that $\pi_2 \wedge \kappa^2 = 0$, $\Omega \wedge \nu_3 = 0$.
2. Analogous to 1 starting from $\kappa \wedge J\Omega = 0$.
3. This formula is a consequence of formula (2.9) together with the definition of J. We have

$$\begin{aligned} 0 &= (*d\Omega) \wedge \Omega - J\Omega \wedge *dJ\Omega \\ &= *(\pi_1 \wedge \Omega) \wedge \Omega - J\Omega \wedge *(\sigma_1 \wedge \Omega) \\ &= -J(\bigstar(\pi_1 \wedge \Omega) \wedge J\Omega) - J(\Omega \wedge \bigstar(\sigma_1 \wedge \Omega)) \\ &= J(J\Omega \wedge \bigstar(\Omega \wedge \pi_1)) + J(\Omega \wedge \bigstar(\Omega \wedge \sigma_1)). \end{aligned}$$

Applying the definition of J and Remark 2.7 we get

$$J(-2\bigstar\pi_1) - J(2J\bigstar\sigma_1) = -2J\bigstar\pi_1 + 2\bigstar\sigma_1 = 0,$$

i.e.

$$\sigma_1 = J\pi_1. \quad \square$$

Hence we can rewrite (2.8) as:

$$\mathrm{d}\kappa = -\frac{3}{2}\sigma_0\Omega + \frac{3}{2}\pi_0 J\Omega + \nu_1 \wedge \kappa + \nu_3;$$
$$\mathrm{d}\Omega = \pi_0\kappa^2 + \pi_1 \wedge \Omega - \pi_2 \wedge \kappa;$$
$$\mathrm{d}J\Omega = \sigma_0\kappa^2 + J\pi_1 \wedge \Omega - \sigma_2 \wedge \kappa.$$

Definition 2.10. The forms $\{\pi_0, \sigma_0, \pi_1, \nu_1, \sigma_2, \nu_3\}$ are called the *torsion forms* of the SU(3)-structure.

A SU(3)-structure is integrable if and only if all of the torsion forms vanish identically.
Several interesting special SU(3)-structures can be described in terms of torsion forms.

1. 6-*dimensional GCY structures*. Let (M, κ, Ω) be a 6-dimensional GCY manifold. The equation $\mathrm{d}\kappa = 0$ implies

$$\pi_0 = \sigma_0 = 0, \qquad \nu_1 = 0, \qquad \nu_3 = 0.$$

Therefore $\mathrm{d}\Omega$ and $\mathrm{d}J\Omega$ reduce to

$$\mathrm{d}\Omega = \pi_1 \wedge \Omega - \pi_2 \wedge \kappa,$$
$$\mathrm{d}J\Omega = J\pi_1 \wedge \Omega - \sigma_2 \wedge \kappa.$$

Since the complex volume form ε associated with (κ, Ω) is of type $(3, 0)$, $\overline{\partial}_J\varepsilon$ is the $(3, 1)$-part (and hence the J anti-invariant part) of $\mathrm{d}\varepsilon$. Thus we have

$$\overline{\partial}_J\varepsilon = \frac{1}{2}(\mathrm{d}\varepsilon - J\mathrm{d}\varepsilon).$$

Thus

$$\begin{aligned}\overline{\partial}_J\varepsilon &= \frac{1}{2}(\mathrm{d}\varepsilon - J\mathrm{d}\varepsilon)\\ &= \frac{1}{2}(\mathrm{d}\Omega + \mathrm{i}\mathrm{d}J\Omega - J\mathrm{d}\Omega - \mathrm{i}J\mathrm{d}J\Omega)\\ &= \frac{1}{2}\{\mathrm{d}\Omega - J\mathrm{d}\Omega + \mathrm{i}(\mathrm{d}J\Omega - J\mathrm{d}J\Omega)\}\\ &= \frac{1}{2}\{\pi_1 \wedge \Omega - J(\pi_1 \wedge \Omega) + \mathrm{i}(J\pi_1 \wedge \Omega - J(J\pi_1 \wedge \Omega))\}\\ &= \pi_1 \wedge \Omega + \mathrm{i}\, J\pi_1 \wedge \Omega.\end{aligned}$$

Hence by Lemma 2.3 the equation $\overline{\partial}_J\varepsilon = 0$ is equivalent to $\pi_1 = 0$. It follows that 6-dimensional GCY structures can be defined as SU(3)-structures satisfying

$$\pi_0 = \sigma_0 = 0, \qquad \nu_1 = \pi_1 = 0, \qquad \nu_3 = 0.$$

2. *Special generalized Calabi–Yau structure*. These structures were introduced and studied first by P. de Bartolomeis in [16].

Definition 2.11. Let M be a 6-dimensional manifold. A special generalized Calabi–Yau structure (SGCY) on M is a SU(3)-structure such that the defining forms κ, Ω are closed, i.e.

$$\mathrm{d}\kappa = 0, \qquad \mathrm{d}\Omega = 0.$$

Special generalized Calabi–Yau manifolds can be considered as a subclass of generalized Calabi–Yau manifold, in fact it is immediately verified that in this case the complex volume form ε associated with (κ, Ω) satisfies the condition 2 of Definition 1.2 (see [18]). SGCY manifolds are taken into consideration also in [8,15,25].
Such a structure can be characterized by

$$\pi_0 = \sigma_0 = 0, \qquad \nu_1 = \pi_1 = 0, \qquad \pi_2 = 0, \qquad \nu_3 = 0.$$

3. *Half-flat structure*. Half-flat manifolds have a central role in the evolution theory developed by Hitchin in [22] and can be used to construct non-compact examples of G_2-manifolds.

Definition 2.12. A SU(3)-structure (κ, Ω) is said to be *half-flat* if the structure forms satisfy the equations

$$d(\kappa \wedge \kappa) = 0, \qquad d\Omega = 0.$$

Let (κ, Ω) be a half-flat structure. By the hypothesis $d\Omega = 0$ we get

$$\pi_i = 0, \quad i = 0, 1, 2;$$

then

$$d\kappa = -\frac{3}{2}\sigma_0 \Omega + \nu_1 \wedge \kappa + \nu_3.$$

On the other hand the hypothesis $d(\kappa \wedge \kappa) = 0$ implies

$$0 = d\kappa \wedge \kappa = -\frac{3}{2}\sigma_0 \Omega \wedge \kappa + \nu_1 \wedge \kappa^2 + \nu_3 \wedge \kappa = \nu_1 \wedge \kappa^2,$$

which forces ν_1 to vanish, since the exterior multiplication by κ^2 is an isomorphism on $\Lambda^1 M$. Therefore half-flat structures can be described as SU(3)-structures satisfying

$$\pi_i = 0, \quad i = 0, 1, 2, \qquad \nu_1 = 0.$$

2.6. Some SU(3) representation theory

Every irreducible representation ρ of the simple Lie group SU(3) can be labeled with a pair of integers (p, q) that represent the highest weight of ρ with respect to a fixed base of the root system of a fixed maximal torus of SU(3). We will denote ρ by $\lambda_{p,q}$. Nevertheless in the sequel we need to deal with *real* representation of SU(3), so (like in [23]) we will define the irreducible real representations $V_{p,q}$ $(p \neq q)$ and $V_{p,p}$ by

$$V_{p,q} \otimes_{\mathbb{R}} \mathbb{C} = \lambda_{p,q} \oplus \lambda_{q,p},$$
$$V_{p,p} \otimes_{\mathbb{R}} \mathbb{C} = \lambda_{p,p}.$$

Keeping this fact in mind, we can use the complex representation theory to decompose a given real SU(3)-representation into irreducible real SU(3)-modules. As is well known (see [10]) the polynomial pointwise invariants of order k are polynomials in a canonically defined section of the vector bundle

$$\mathcal{Q} \times_{\rho_1 \times \cdots \times \rho_k} (V_1(\mathfrak{su}(3)) \oplus \cdots \oplus V_k(\mathfrak{su}(3))),$$

where $\mathcal{Q}$ is the SU(3)-reduction and $V_j(\mathfrak{su}(3))$ is the SU(3)-representation uniquely defined by

$$(\mathfrak{gl}(6, \mathbb{R})/\mathfrak{su}(3)) \otimes S^j(\mathbb{R}^6) = V_j(\mathfrak{su}(3)) \oplus (\mathbb{R}^6 \otimes S^{j+1}(\mathbb{R}^6)).$$

For the first-order invariants we have

$$V_1(\mathfrak{su}(3)) = \mathfrak{so}(6)/\mathfrak{su}(3) \otimes \mathbb{R}^6$$

so that

$$V_1(\mathfrak{su}(3)) = 2V_{0,0} \oplus 2(\mathbb{R}^6)^* \oplus 2\Lambda^2_8 \oplus \Lambda^3_{12}$$

which matches with the degree and types of our torsion forms. Rather standard calculation in $\mathfrak{su}(3)$-representation theory allows us to decompose also the 252-dimensional representation $V_2(\mathfrak{su}(3))$ into $\mathfrak{su}(3)$-irreducible submodules

$$V_2(\mathfrak{su}(3)) = 3V_{0,0} \oplus 4V_{1,0} \oplus 5V_{1,1} \oplus 3V_{2,1} \oplus 4V_{2,0} \oplus V_{3,0} \oplus V_{2,2}.$$

3. Riemannian invariants of SU(3)-structures

3.1. The Levi-Civita connection

Fix a SU(3)-reduction $\mathcal{Q}$ of the linear frame bundle $\mathcal{L}(M)$, given by the pair (κ, Ω). $\mathcal{Q}$ is a subbundle of the principal SO(6)-bundle $p : \mathcal{F} \to M$ of the normal frames of the metric g associated with the pair (κ, Ω). Consider on the bundle $\mathcal{F}$ the tautological $\mathbb{R}^6$-valued 1-form ω defined by $\omega[u](v) = u(p_*[u]v)$ for every $u \in \mathcal{F}$ and $v \in T_u\mathcal{F}$. On $\mathcal{F}$ we have also the Levi-Civita connection 1-form ψ taking values in $\mathfrak{so}(6)$. Using the canonical basis $\{e_1, \dots, e_6\}$ of $\mathbb{R}^6$ we will regard ω as a vector of $\mathbb{R}$-valued 1-forms on $\mathcal{F}$

$$\omega = \omega_1 e_1 + \cdots + \omega_6 e_6$$

and ψ as a skew-symmetric matrix of 1-forms, i.e. $\psi = (\psi_{ij})$. With this notation the first structure equation relating ω and ψ

$$\mathrm{d}\omega = -\psi \wedge \omega, \tag{3.1}$$

becomes $\mathrm{d}\omega_i = -\psi_{ij} \wedge \omega_j$. Note that Eq. (3.1) simply means that ψ is torsion-free.
The curvature of ψ is by definition the $\mathfrak{so}(6)$-valued 2-form $\Psi = \mathrm{d}\psi + \psi \wedge \psi$. In index notation

$$\Psi_{ij} = \mathrm{d}\psi_{ij} + \psi_{ik} \wedge \psi_{kj} = \frac{1}{2} R_{ijkl}\omega_k \wedge \omega_l.$$

We consider the pull-backs of ψ and ω to $\mathcal{Q}$ and denote them by the same symbols for the sake of brevity. The intrinsic torsion of the SU(3)-structure measures the failing of ψ to take values in $\mathfrak{su}(3)$. More precisely, according to the splitting $\mathfrak{so}(6) = \mathfrak{su}(3) \oplus [\mathbb{R}]_1 \oplus [\mathbb{R}^6]_2$, we decompose ψ as follows

$$\psi = \theta + [\mu]_1 + [\tau]_2.$$

Thus θ is a connection 1-form on $\mathcal{Q}$ which in general is not torsion-free.
As before we shall regard τ as a vector of 1-forms $\tau = \tau_i e_i$. Furthermore we can write

$$\tau_i = T_{ij}\omega_j \quad \text{and} \quad \mu = M_i\omega_i, \tag{3.2}$$

where T_{ij} and M_i are smooth functions. The fact that ψ is torsion-free implies

$$\mathrm{d}\omega_i = -\theta_{ij} \wedge \omega_j - \epsilon_{ijk}\tau_k \wedge \omega_j - \kappa_{ij}\mu \wedge \omega_j. \tag{3.3}$$

3.2. *The curvature in index notation*

In order to decompose the curvature 2-form we give the following

Lemma 3.1. *These identities hold:*

1. $\theta \wedge [\mu]_1 + [\mu]_1 \wedge \theta = 0$;
2. $[\tau]_2 \wedge [\mu]_1 - [\mu]_1 \wedge [\tau]_2 = 0$;
3. $\theta \wedge [\tau]_2 + [\tau]_2 \wedge \theta = [\theta \wedge \tau]_2$;
4. $[\tau]_2 \wedge [\mu]_1 + [[\mu]_1 \wedge \tau]_2 = 0$.

Proof. The proof is a straightforward application of ϵ-identities (2.6). To see how things work, we prove the first one. Since θ takes values in $\mathfrak{su}(3)$ we have

$$\epsilon_{pkl}\theta_{kl} = \epsilon_{klp}\theta_{kl} = 0.$$

So

$$\bar{\epsilon}_{ijp}\epsilon_{klp}\theta_{kl} = 0$$

for every $i, j = 1, \dots, 6$. Then applying the ϵ-identities (2.6) we get

$$\begin{aligned} 0 &= \bar{\epsilon}_{ijp}\epsilon_{klp}\theta_{kl} \\ &= (-\kappa_{ik}\delta_{jl} + \kappa_{jk}\delta_{il} + \kappa_{il}\delta_{jk} - \kappa_{jl}\delta_{ik})\theta_{kl} \\ &= 2\kappa_{jk}\theta_{ki} - 2\kappa_{ik}\theta_{kj}, \end{aligned}$$

i.e.

$$\kappa_{jk}\theta_{ki} = \kappa_{ik}\theta_{kj}.$$

Consequently

$$\theta_{ik} \wedge \kappa_{kj}\mu + \kappa_{ik}\mu \wedge \theta_{kj} = 0,$$

i.e.

$$\theta \wedge [\mu]_1 + [\mu]_1 \wedge \theta = 0. \quad \square$$

Now we can introduce the following quantities

$$D\theta = \mathrm{d}\theta + \theta \wedge \theta + [\tau]_2 \wedge [\tau]_2 - \frac{2}{3}[\kappa_{ij}\tau_i \wedge \tau_j]_1, \tag{3.4}$$

$$D\tau = \mathrm{d}\tau + \theta \wedge \tau - 2[\mu]_1 \wedge \tau, \tag{3.5}$$

$$D\mu = \mathrm{d}\mu + \frac{2}{3}\kappa_{ij}\tau_i \wedge \tau_j. \tag{3.6}$$

With this definition $D\theta$ takes values in $\mathfrak{su}(3)$. Moreover by Lemma 3.1 we get

$$\begin{aligned}\Psi &= \mathrm{d}(\theta + [\tau]_2 + [\mu]_1) + (\theta + [\tau]_2 + [\mu]_1) \wedge (\theta + [\tau]_2 + [\mu]_1)\\ &= D\theta + [D\tau]_2 + [D\mu]_1.\end{aligned}$$

Using the ω-frame we shall write

$$D\theta_{ij} = \frac{1}{2}S_{ijkl}\omega_k \wedge \omega_l, \tag{3.7}$$

$$D\tau_i = \frac{1}{2}T_{ijk}\omega_j \wedge \omega_k, \tag{3.8}$$

$$D\mu = \frac{1}{2}N_{kl}\omega_k \wedge \omega_l. \tag{3.9}$$

By the definition of the curvature form we have

$$R_{ijkl} = S_{ijkl} + \epsilon_{ijp}T_{pkl} + \kappa_{ij}N_{kl}.$$

In this notation the first Bianchi identity

$$\Psi \wedge \omega = 0,$$

has the indicial expression

$$S_{ijkl} + S_{iljk} + S_{iklj} + \epsilon_{ijp}T_{pkl} + \epsilon_{ilp}T_{pjk} + \epsilon_{ikp}T_{plj} + \kappa_{ij}N_{kl} + \kappa_{il}N_{jk} + \kappa_{ik}N_{lj} = 0 \tag{3.10}$$

Let $\mathrm{Ric}_{ij} = R_{ikkj}$ and $s = \mathrm{Ric}_{kk}$ be respectively the Ricci tensor and the scalar curvature of (M, g). Starting from Eq. (3.10) a long, but straightforward computation gives the following

Theorem 3.2. *In the previous notation we have*

$$\begin{aligned}\mathrm{Ric}_{ij} &= 2\epsilon_{ipq}T_{pqj} - 3\kappa_{ip}N_{pj},\\ s &= 2\epsilon_{kpq}T_{pqk} - 3\kappa_{kp}N_{pk}.\end{aligned}$$

3.3. Ricci tensor in terms of torsion forms

Denote by π the projection $\pi : \mathcal{Q} \to M$. In terms of the ω-frame the pull-backs of the structure forms take their standard expression, i.e.

$$\pi^*(\Omega) = \frac{1}{6}\epsilon_{ijk}\omega_i \wedge \omega_j \wedge \omega_k,$$

$$\pi^*(J\Omega) = \frac{1}{6}\overline{\epsilon}_{ijk}\omega_i \wedge \omega_j \wedge \omega_k,$$

$$\pi^*(\kappa) = \frac{1}{2}\kappa_{ij}\omega_i \wedge \omega_j.$$

Taking into account formula (3.3) and ϵ-identities, we immediately get

Proposition 3.3. *The derivatives of the structure forms are*

$$\begin{aligned}
d\pi^*(\Omega) &= \frac{1}{2}(-\kappa_{ja}\kappa_{kb} + \kappa_{jb}\kappa_{ka})\tau_b \wedge \omega_a \wedge \omega_j \wedge \omega_k - 3\mu \wedge \pi^*(J\Omega),\\
d\pi^*(J\Omega) &= (\tau_j \wedge \omega_j) \wedge \pi^*(\kappa) - 3\mu \wedge \pi^*(\Omega),\\
d\pi^*(\kappa) &= \overline{\epsilon}_{lrj}\tau_l \wedge \omega_r \wedge \omega_j.
\end{aligned}$$

Now we can decompose the derivatives of the structure forms: a direct computation gives the following formulae

$$\begin{aligned}
\pi^*(\pi_0) &= \frac{2}{3}T_{ii},\\
\pi^*(\pi_1) &= \epsilon_{ijk}T_{ij}\omega_k + 3\kappa_{ik}M_i\omega_k,\\
\pi^*(\pi_2) &= \frac{1}{2}\overline{\epsilon}_{sra}\epsilon_{aij}T_{sr}\omega_i \wedge \omega_j - 2\kappa_{ia}T_{aj}\omega_i \wedge \omega_j + \frac{2}{3}T_{ii}\pi^*(\kappa),\\
\pi^*(\sigma_0) &= \frac{2}{3}\kappa_{ij}T_{ij},\\
\pi^*(\sigma_2) &= \frac{1}{2}\epsilon_{rsa}\epsilon_{aij}T_{rs}\omega_i \wedge \omega_j - 2T_{ij}\omega_i \wedge \omega_j + \frac{2}{3}\kappa_{ij}T_{ij}\pi^*(\kappa),\\
\pi^*(\nu_1) &= \epsilon_{ijk}T_{ij}\omega_k,\\
\pi^*(\nu_3) &= \overline{\epsilon}_{aij}T_{ak}\omega_i \wedge \omega_j \wedge \omega_k + \frac{1}{6}\kappa_{ab}T_{ab}\epsilon_{ijk}\omega_i \wedge \omega_j \wedge \omega_k\\
&\quad - \frac{1}{6}T_{aa}\overline{\epsilon}_{ijk}\omega_i \wedge \omega_j \wedge \omega_k - \frac{1}{2}T_{ab}\epsilon_{abi}\kappa_{jk}\omega_i \wedge \omega_j \wedge \omega_k.
\end{aligned}$$

Warning: From now on we identify the torsion forms with their pull-backs to the principal SU(3)-bundle $\mathcal{Q}$. Combining the previous formulae and (3.3) we are able to prove the following (see the Appendix)

Theorem 3.4. *In terms of torsion forms the scalar curvature of the metric induced by the* SU(3)*-structure is expressed as*

$$s = \frac{15}{2}\pi_0^2 + \frac{15}{2}\sigma_0^2 + 2d^*\pi_1 + 2d^*\nu_1 - |\nu_1|^2 - \frac{1}{2}|\sigma_2|^2 - \frac{1}{2}|\pi_2|^2 - \frac{1}{2}|\nu_3|^2 + 4\langle \pi_1, \nu_1\rangle. \quad (3.11)$$

Here we collect some consequences of formula (3.11) when the SU(3)-structure has special features.

1. *GCY structure*. The condition $\overline{\partial}_J\epsilon = 0$ reads as $\pi_1 = 0$ (see Section 2.5), so that, taking into account $d\kappa = 0$,
$$s = -\frac{1}{2}|\sigma_2|^2 - \frac{1}{2}|\pi_2|^2.$$
2. *SGCY structure*. This is a special case of the previous one with the extra condition $\pi_2 = 0$. The scalar curvature takes the form
$$s = -\frac{1}{2}|\sigma_2|^2. \quad (3.12)$$
3. *Half-flat structure*. The condition $d\kappa \wedge \kappa = 0$ reads in terms of torsion forms as $\nu_1 = 0$. Thus in the half-flat case the scalar curvature takes the form
$$s = \frac{15}{2}\sigma_0^2 - \frac{1}{2}|\sigma_2|^2 - \frac{1}{2}|\nu_3|^2.$$

Corollary 3.5. *The scalar curvature of a* 6*-dimensional generalized Calabi–Yau manifold is everywhere non-positive and it vanishes identically if and only if the* SU(3)*-structure has no torsion.*

Now we write the Ricci curvature $\mathrm{Ric}_{ij} = 2\epsilon_{ipq}T_{pqj} - 3\kappa_{ip}N_{pj}$ in terms of the torsion forms using the operators ι and γ defined in Section 2.3.

Theorem 3.6. *If M is endowed with the* SU(3)*-structure (κ, Ω) with torsion forms given by* (2.8)*, then the traceless part of the Ricci tensor of the induced metric is*

$$\mathrm{Ric}_0 = \iota^{-1}(E_1(\phi_1)) + \gamma^{-1}(E_2(\phi_2)), \tag{3.13}$$

where

$$\begin{aligned}\phi_1 &= -*(\nu_1 \wedge J\nu_3) + \frac{1}{4}*(\pi_2 \wedge \pi_2) + \frac{1}{4}*(\sigma_2 \wedge \sigma_2)\\ &\quad + \mathrm{d}J\pi_1 + \frac{1}{2}\mathrm{d}^*\nu_3 + \frac{1}{2}\mathrm{d}^*(\nu_1 \wedge \kappa) - \frac{1}{4}\mathrm{d}*(\pi_0\Omega) + \frac{1}{4}\mathrm{d}^*(\sigma_0\Omega),\\ \phi_2 &= -2\sigma_0\nu_3 - 4\sigma_2 \wedge \nu_1 - 2Jd\pi_2 - 2\bigstar\mathrm{d}\sigma_2 - 4\mathrm{d}*(\nu_1 \wedge *\Omega) + -2\mathrm{d}*(J\pi_1 \wedge \Omega) + 2\pi_0 J\nu_3\\ &\quad - 2J\mathrm{d}*(\pi_1 \wedge \Omega) - 4\pi_2 \wedge J\pi_1 + 4\nu_1 \wedge *(J\pi_1 \wedge \Omega) - 2J\nu_1 \wedge *(\nu_1 \wedge \Omega) - \frac{1}{2}Q(\nu_3, \nu_3),\end{aligned}$$

E_1 and E_2 are the maps defined by Eqs. (2.4) and (2.5) and Q is the bilinear form $Q : \Lambda^3_{12}M \times \Lambda^3_{12}M \to \Lambda^3 M$ defined by

$$Q(\alpha, \beta) = \epsilon_{ijl}\iota_{e_j}\iota_{e_i}\alpha \wedge \iota_{e_l}\beta,$$

where $\{e_1, \ldots, e_6\}$ is a unitary frame and ι denotes the contraction of forms.

Remark 3.7. The formulae for the scalar curvature and for the traceless part of the Ricci tensor are justified by representation theory. Both s and Ric_0 must be linear combinations of linear terms in $V_2(\mathfrak{su}(3))$ and quadratic terms in $V_1(\mathfrak{su}(3))$. For the scalar curvature the terms must take values in the $V_{0,0}$ copies of V_1 and V_2, while for the Ricci curvature the terms must take values in Λ^2_8 and Λ^3_{12} copies of V_1 and V_2 (for $S^2_0 = \Lambda^2_8 \oplus \Lambda^3_{12}$). So we have to consider:

$$S^2(V_1(\mathfrak{su}(3))) = 11V_{0,0} \oplus 13V_{1,0} \oplus 17V_{1,1} \oplus 12V_{2,0} \oplus 3V_{3,0} \oplus 4V_{2,2} \oplus 9V_{2,1} \oplus 2V_{3,1}.$$

The 11 copies of $V_{0,0}$ are generated by

- $\pi_0^2, \sigma_0^2, \pi_0\sigma_0$;
- $|\pi_1|^2, |\nu_1|^2, \langle\pi_1, \nu_1\rangle$ and another bilinear expression in π_1, ν_1 which does not appear in formula (3.11);
- $|\sigma_2|^2, |\pi_2|^2$, and a bilinear expression in π_2, σ_2 which does not appear;
- $|\nu_3|^2$.

The 17 copies of $V_{1,1}$ are generated by the projections of

- $\pi_0\pi_2, \pi_0\sigma_2, \sigma_0\sigma_2, \sigma_0\pi_2$;
- four bilinear expressions in π_1 and ν_1 which do not appear in formula (3.13);
- $*\pi_1 \wedge J\nu_3$ and three more bilinear expressions in π_1 and ν_3;
- $*(\pi_2 \wedge \pi_2), *(\sigma_2 \wedge \sigma_2)$ and two more bilinear expressions in π_2 and σ_2;
- a bilinear form in ν_3.

The 12 copies of $V_{2,0}$ are generated by the projections of

- $\pi_0\nu_3, \sigma_0\nu_3$;
- $\nu_1 \wedge *(J\pi_1 \wedge \Omega), J\nu_1 \wedge *(\nu_1 \wedge \Omega)$ and another two bilinear expressions in π_1, ν_1;
- $\sigma_2 \wedge \nu_1, \pi_2 \wedge \nu_1, \sigma_2 \wedge \pi_1, \pi_2 \wedge \pi_1$;
- two bilinear expressions in σ_2, ν_3 and π_2, ν_3;
- $Q(\nu_3, \nu_3)$.

An analogous discussion can be given for the second-order expressions after considering the splitting:

$$V_2(\mathfrak{su}(3)) = 3V_{0,0} \oplus 4V_{1,0} \oplus 5V_{1,1} \oplus 3V_{2,1} \oplus 4V_{2,0} \oplus V_{3,0} \oplus V_{2,2}.$$

4. The Ricci tensor in the GCY case

Suppose now that the pair (κ, Ω) gives a generalized Calabi–Yau structure on M. In this case all the torsion is encoded by π_2 and σ_2; in fact $\mathrm{d}\Omega$ and $\mathrm{d}J\Omega$ reduce to

$$\mathrm{d}\Omega = -\pi_2 \wedge \kappa, \qquad \mathrm{d}J\Omega = -\sigma_2 \wedge \kappa.$$

Therefore we get

$$0 = \mathrm{d}^2\Omega = -\mathrm{d}\pi_2 \wedge \kappa,$$
$$0 = \mathrm{d}^2 J\Omega = -\mathrm{d}\sigma_2 \wedge \kappa,$$

i.e. $\mathrm{d}\pi_2$ and $\mathrm{d}\sigma_2$ are effective 3-forms. Since $\pi_2 \in \Lambda^2_8 M$

$$\begin{aligned} 0 = \mathrm{d}(\pi_2 \wedge \Omega) &= \mathrm{d}\pi_2 \wedge \Omega + \pi_2 \wedge \mathrm{d}\Omega \\ &= \mathrm{d}\pi_2 \wedge \Omega - \pi_2 \wedge \pi_2 \wedge \kappa \\ &= \mathrm{d}\pi_2 \wedge \Omega + \pi_2 \wedge *\pi_2 \\ &= \mathrm{d}\pi_2 \wedge \Omega + |\pi_2|^2 * 1, \end{aligned}$$

i.e.

$$\mathrm{d}\pi_2 \wedge \Omega = -|\pi_2|^2 * 1.$$

Analogously we get

$$\mathrm{d}\sigma_2 \wedge J\Omega = -|\sigma_2|^2 * 1.$$

Now we can express the Ricci tensor of a generalized Calabi–Yau manifold in terms of π_2 and σ_2. In this case Eq. (3.13) reduces to

$$\mathrm{Ric}_0 = \frac{1}{4}\iota^{-1}(E_1(*(\pi_2 \wedge \pi_2 + \sigma_2 \wedge \sigma_2))) - 2\gamma^{-1}(E_2(J\mathrm{d}\pi_2 + \bigstar\mathrm{d}\sigma_2)).$$

Since $\mathrm{d}\sigma_2$ is effective, $\bigstar\mathrm{d}\sigma_2 = -\mathrm{d}\sigma_2$. Thus

$$\mathrm{Ric}_0 = \frac{1}{4}\iota^{-1}(E_1(*(\pi_2 \wedge \pi_2 + \sigma_2 \wedge \sigma_2))) - 2\gamma^{-1}(E_2(J\mathrm{d}\pi_2 - \mathrm{d}\sigma_2)).$$

By the definitions of E_1 and E_2, using the J-invariance of π_2 and formula (2.3), we have

$$\begin{aligned} E_1(*(\pi_2 \wedge \pi_2)) &= *(\pi_2 \wedge \pi_2) - \frac{1}{9} * ((\pi_2 \wedge \pi_2 + *(\pi_2 \wedge \pi_2) \wedge \kappa) \wedge \kappa)\kappa \\ &= *(\pi_2 \wedge \pi_2) + \frac{1}{9}|\pi_2|^2\kappa - \frac{1}{9} * (*(\pi_2 \wedge \pi_2) \wedge \kappa^2)\kappa \\ &= *(\pi_2 \wedge \pi_2) + \frac{1}{9}|\pi_2|^2\kappa + \frac{2}{9}|\pi_2|^2\kappa \\ &= *(\pi_2 \wedge \pi_2) + \frac{1}{3}|\pi_2|^2\kappa \end{aligned}$$

and

$$\begin{aligned} E_2(\mathrm{d}\pi_2) &= \mathrm{d}\pi_2 - \frac{1}{2} * (J\mathrm{d}\pi_2 \wedge \kappa) \wedge \kappa - \frac{1}{4} * (\mathrm{d}\pi_2 \wedge J\Omega)\Omega + \frac{1}{4} * (\mathrm{d}\pi_2 \wedge \Omega)J\Omega \\ &= \mathrm{d}\pi_2 - \frac{1}{4} * (\mathrm{d}\pi_2 \wedge J\Omega)\Omega - \frac{1}{4}|\pi_2|^2 J\Omega \\ &= \mathrm{d}\pi_2 + \frac{1}{4} * (\pi_2 \wedge \sigma_2 \wedge \kappa)\Omega - \frac{1}{4}|\pi_2|^2 J\Omega, \end{aligned}$$

where in the last step we have used

$$0 = \mathrm{d}(\pi_2 \wedge J\Omega) = \mathrm{d}\pi_2 \wedge J\Omega + \pi_2 \wedge \mathrm{d}J\Omega = \mathrm{d}\pi_2 \wedge J\Omega - \pi_2 \wedge \sigma_2 \wedge \kappa.$$

In the same way we get

$$E_1(*(\sigma_2 \wedge \sigma_2)) = *(\sigma_2 \wedge \sigma_2) + \frac{1}{3}|\sigma_2|^2\kappa$$

and

$$E_2(d\sigma_2) = d\sigma_2 + \frac{1}{4} * (\pi_2 \wedge \sigma_2 \wedge \kappa) J\Omega + \frac{1}{4}|\sigma_2|^2 \Omega.$$

Therefore, taking into account that E_2 commutes with J, the traceless Ricci tensor of a generalized Calabi–Yau manifold is given by

$$\mathrm{Ric}_0 = \frac{1}{4}\iota^{-1}(*(\sigma_2 \wedge \sigma_2 + \pi_2 \wedge \pi_2) + \frac{1}{3}(|\sigma_2|^2 + |\pi_2|^2)\kappa) - 2\gamma^{-1}(J d\pi_2 - d\sigma_2 + \frac{1}{4}(|\pi_2|^2 - |\sigma_2|^2)\Omega). \quad (4.1)$$

Formula (4.1) implies that the metric induced by a GCY structure (κ, Ω) is Einstein (*i.e.* $\mathrm{Ric}_0 = 0$) if and only if the torsion forms π_2, σ_2 satisfy

$$\begin{cases} \sigma_2 \wedge \sigma_2 + \pi_2 \wedge \pi_2 + \dfrac{1}{6}(|\pi_2|^2 + |\sigma_2|^2)\kappa \wedge \kappa = 0 \\ J d\pi_2 - d\sigma_2 + \dfrac{1}{4}(|\pi_2|^2 - |\sigma_2|^2)\Omega = 0. \end{cases} \quad (4.2)$$

In the special case of SGCY manifolds we can prove

Corollary 4.1. *A 6-dimensionals* SGCY *manifold is Einstein if and only if it is a genuine Calabi–Yau manifold.*

The proof of Corollary 4.1 relies on the following lemma which is interesting in its own right.

Lemma 4.2. *Let (V, κ, Ω) be a 6-dimensional symplectic vector space endowed with a normalized κ-positive 3-form. If $\alpha \neq 0$ belongs to $\Lambda^2_8 V^*$, then $\alpha \wedge \alpha$ does not belong to the 1-dimensional* SU(3)*-module generated by $\kappa \wedge \kappa$.*

Proof. The key observation here is that $\Lambda^2_8 V^*$ is isomorphic as a SU(3)-representation to the adjoint representation $V_{1,1}$. Since every element in $\mathfrak{su}(3)$ is Ad(SU(3))-conjugate to an element of a fixed Cartan subalgebra of $\mathfrak{su}(3)$, there exists a SU(3)-basis $\{e^1, \ldots, e^6\}$ of V^* such that

$$\alpha = \lambda_1 e^{12} + \lambda_2 e^{34} - (\lambda_1 + \lambda_2)e^{56},$$

for some $\lambda_1, \lambda_2 \in \mathbb{R}$. Now suppose that $\alpha \wedge \alpha = q\kappa \wedge \kappa$ for some $q \in \mathbb{R}$. Setting to zero the three components of $\alpha \wedge \alpha - q\kappa \wedge \kappa$ gives the equations

$$\lambda_1^2 + \lambda_1\lambda_2 + q = 0,$$
$$\lambda_2^2 + \lambda_1\lambda_2 + q = 0,$$
$$\lambda_1\lambda_2 - q = 0,$$

which readily imply $q = 0$. □

Proof of Corollary 4.1. Since in the SGCY case $\pi_2 = 0$, taking into account Lemma 4.2, the first equation of (4.2) can be satisfied if and only if $|\sigma_2|^2 = 0$. Therefore the Einstein condition forces (κ, Ω) to be a Calabi–Yau structure on M. □

Remark 4.3. In [19] it has been proven (see Theorem 1) that a *compact* Einstein almost Kähler manifold with vanishing first Chern class is actually a Kähler–Einstein manifold. Note that our result holds with no compactness assumption.

5. An explicit example

In this last section we carry out the computation of the Ricci tensor and the intrinsic torsion of a left-invariant SU(3)-structure on a particular 6-dimensional nilmanifold.

Let G be the nilpotent Lie group of the matrices of the form

$$A = \begin{pmatrix} 1 & 0 & x_1 & x_3 & 0 & 0 \\ 0 & 1 & x_2 & x_4 & 0 & 0 \\ 0 & 0 & 1 & x_5 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & x_6 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

where $x_1, x_2, x_3, x_4, x_5, x_6$ are real numbers. Let Γ be the set of matrices in G having integral entries; then $M := \mathrm{G}/\Gamma$ is a compact parallelizable smooth manifold. Let $\{X_1, \dots, X_n\}$ be the global frame on M given by

$$X_1 = \frac{\partial}{\partial x_5} + x_1 \frac{\partial}{\partial x_3} + x_2 \frac{\partial}{\partial x_4}, \qquad X_2 = \frac{\partial}{\partial x_6},$$
$$X_3 = \frac{\partial}{\partial x_2}, \qquad X_4 = \frac{\partial}{\partial x_3}, \qquad X_5 = \frac{\partial}{\partial x_1}, \qquad X_6 = \frac{\partial}{\partial x_4}.$$

We have that

$$[X_1, X_3] = -X_6, \qquad [X_1, X_5] = -X_4$$

and the other brackets are zero. Let $\{\alpha_1, \dots, \alpha_6\}$ be the dual frame of $\{X_1, \dots, X_n\}$; then

$$\begin{cases} \mathrm{d}\alpha_1 = \mathrm{d}\alpha_2 = \mathrm{d}\alpha_3 = \mathrm{d}\alpha_5 = 0 \\ \mathrm{d}\alpha_4 = \alpha_{15} \\ \mathrm{d}\alpha_6 = \alpha_{13}. \end{cases}$$

Therefore the *closed* global forms

$$\kappa = \alpha_{12} + \alpha_{34} + \alpha_{56},$$
$$\Omega = \alpha_{135} - \alpha_{146} - \alpha_{245} - \alpha_{236}$$

define a SGCY structure on M. Let J be the almost complex structure on M induced by the SU(3)-structure; then on the frame $\{X_1, \dots, X_6\}$ one has

$$J(X_1) = X_2, \qquad J(X_3) = X_4, \qquad J(X_5) = X_6.$$

We have

$$\mathrm{d}J\Omega = \mathrm{d}(-\alpha_{246} + \alpha_{235} + \alpha_{145} + \alpha_{136}) = \alpha_{1234} - \alpha_{1256} = (\alpha_{34} - \alpha_{56}) \wedge \kappa,$$

i.e., with the notation of (2.8),

$$\sigma_2 = \alpha_{56} - \alpha_{34}.$$

Since (M, κ, Ω) is a SGCY manifold, σ_2 is the only non-zero torsion form.

Note that the metric associated with (κ, Ω) is

$$g = \sum_{i=1}^{n} \alpha_i \otimes \alpha_i.$$

Consequently we have $|\sigma_2|^2 = 2$, and hence formula (3.12) implies $s = -1$.

Using (4.1) we can compute the Ricci tensor of g: we have

$$\begin{aligned} \mathrm{Ric}_0 &= \iota^{-1}\left(-\frac{1}{2}\alpha_{12} + \frac{1}{6}\kappa\right) + \gamma^{-1}(-4\alpha_{135} + \Omega) \\ &= \iota^{-1}\left(-\frac{1}{3}\alpha_{12} + \frac{1}{6}\alpha_{34} + \frac{1}{6}\alpha_{56}\right) + \gamma^{-1}(-3\alpha_{135} - \alpha_{146} - \alpha_{245} - \alpha_{236}). \end{aligned}$$

Let ∇ be the Levi-Civita connection of g; then

$$
\begin{aligned}
&\nabla_1 X_3 = -\frac{1}{2}X_6, \qquad \nabla_1 X_6 = \frac{1}{2}X_3, \qquad \nabla_3 X_6 = -\frac{1}{2}X_1,\\
&\nabla_3 X_1 = \frac{1}{2}X_6, \qquad \nabla_6 X_1 = \frac{1}{2}X_3, \qquad \nabla_6 X_3 = -\frac{1}{2}X_1,\\
&\nabla_1 X_5 = -\frac{1}{2}X_4, \qquad \nabla_1 X_4 = \frac{1}{2}X_5, \qquad \nabla_5 X_4 = -\frac{1}{2}X_1,\\
&\nabla_5 X_1 = \frac{1}{2}X_4, \qquad \nabla_4 X_1 = \frac{1}{2}X_5, \qquad \nabla_4 X_5 = -\frac{1}{2}X_1,
\end{aligned}
$$

where $\nabla_i X_j$ stands for $\nabla_{X_i} X_j$. Now are ready to compute the torsion of this SU(3)-manifold. We immediately have

$$
\psi = \frac{1}{2}\begin{pmatrix}
0 & 0 & -\alpha_6 & -\alpha_5 & -\alpha_4 & -\alpha_3\\
0 & 0 & 0 & 0 & 0 & 0\\
\alpha_6 & 0 & 0 & 0 & 0 & \alpha_1\\
\alpha_5 & 0 & 0 & 0 & -\alpha_1 & 0\\
\alpha_4 & 0 & 0 & \alpha_1 & 0 & 0\\
\alpha_3 & 0 & -\alpha_1 & 0 & 0 & 0
\end{pmatrix}
$$

and a computation gives

$$
\theta = \frac{1}{4}\begin{pmatrix}
0 & 0 & -\alpha_6 & -\alpha_5 & -\alpha_4 & -\alpha_3\\
0 & 0 & \alpha_5 & -\alpha_6 & \alpha_3 & -\alpha_4\\
\alpha_6 & -\alpha_5 & 0 & 0 & 0 & 2\alpha_1\\
\alpha_5 & \alpha_6 & 0 & 0 & -2\alpha_1 & 0\\
\alpha_4 & -\alpha_3 & 0 & 2\alpha_1 & 0 & 0\\
\alpha_3 & \alpha_4 & -2\alpha_1 & 0 & 0 & 0
\end{pmatrix}
$$

and

$$
\tau = \frac{1}{4}\begin{pmatrix}
0\\ 0\\ \alpha_5\\ -\alpha_3\\ -\alpha_6\\ \alpha_5
\end{pmatrix}, \qquad \mu = 0.
$$

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Appendix

In this appendix we give proofs of Lemma 2.8 and Theorem 3.4.

Proof of Lemma 2.8. Let N be the Riemannian product $N = M \times \mathbb{R}$. Denote by

$$
\begin{aligned}
&p_1 : N \to M,\\
&p_2 : N \to \mathbb{R}
\end{aligned}
$$

the projections. The 3-form

$$\sigma = p_1^*(\Omega) + p_1^*(\kappa) \wedge p_2^*(\mathrm{d}t),$$

defines a G_2-structure on N. From now on we identify the forms κ, Ω, $\mathrm{d}t$ with their respective pull-backs to N. Let us denote by $*_\sigma$ and $*$ the Hodge operator associated with the metric induced by σ and by the SU(3)-structure on M respectively. Thus

$$\begin{aligned}
&\mathrm{d}\sigma = \mathrm{d}\Omega + \mathrm{d}\kappa \wedge \mathrm{d}t,\\
&*_\sigma \sigma = (*\Omega) \wedge \mathrm{d}t + *\kappa = J\Omega \wedge \mathrm{d}t + \frac{1}{2}\kappa^2,\\
&\mathrm{d} *_\sigma \sigma = \mathrm{d}J\Omega \wedge \mathrm{d}t + \mathrm{d}\kappa \wedge \kappa,\\
&*_\sigma \mathrm{d}\sigma = (*\mathrm{d}\Omega) \wedge \mathrm{d}t - *\mathrm{d}\kappa,\\
&*_\sigma \mathrm{d} *_\sigma \sigma = *\mathrm{d}J\Omega + *(\mathrm{d}\kappa \wedge \kappa) \wedge \mathrm{d}t.
\end{aligned}$$

Now we use the formula

$$*_\sigma \sigma \wedge *_\sigma(\mathrm{d} *_\sigma \sigma) + (*_\sigma \mathrm{d}\sigma) \wedge \sigma = 0, \tag{A.1}$$

proved by Bryant in [9]. Now we have

$$\begin{aligned}
*_\sigma \sigma \wedge *_\sigma(\mathrm{d} *_\sigma \sigma) + (*_\sigma \mathrm{d}\sigma) \wedge \sigma = {} & J\Omega \wedge (*\mathrm{d}J\Omega) \wedge \mathrm{d}t + \frac{1}{2}\kappa^2 \wedge (*(\mathrm{d}\kappa \wedge \kappa)) \wedge \mathrm{d}t\\
& + \frac{1}{2}\kappa^2 \wedge *\mathrm{d}J\Omega - (*\mathrm{d}\Omega) \wedge \Omega \wedge \mathrm{d}t - (*\mathrm{d}\kappa) \wedge \Omega - (*\mathrm{d}\kappa) \wedge \kappa \wedge \mathrm{d}t.
\end{aligned}$$

Therefore Eq. (A.1) implies

- $(*\mathrm{d}\kappa) \wedge \Omega = \frac{1}{2}\kappa^2 \wedge *\mathrm{d}J\Omega$, which is indeed an easy consequence of $\Omega \wedge \kappa = 0$;
- $J\Omega \wedge (*\mathrm{d}J\Omega) + \frac{1}{2}\kappa^2 \wedge *(\mathrm{d}\kappa \wedge \kappa) - (*\mathrm{d}\Omega) \wedge \Omega - (*\mathrm{d}\kappa) \wedge \kappa = 0$.

In order to show that Eq. (2.9) holds, we need to prove the following identity

$$\frac{1}{2}\kappa^2 \wedge *(\mathrm{d}\kappa \wedge \kappa) = (*\mathrm{d}\kappa) \wedge \kappa. \tag{A.2}$$

The decomposition of 3-forms on M implies

$$\frac{1}{2}\kappa^2 \wedge *(\mathrm{d}\kappa \wedge \kappa) = \frac{1}{2}\kappa^2 \wedge *(\nu_1 \wedge \kappa^2) = (\bigstar\kappa) \wedge *(\nu_1 \wedge \kappa^2)$$

and

$$(*\mathrm{d}\kappa) \wedge \kappa = (*(\nu_1 \wedge \kappa)) \wedge \kappa,$$

where $\nu_1 \wedge \kappa \in \Lambda^3_6 M = \{\gamma \in \Lambda^3 M \mid \bigstar\gamma = \gamma\}$. Now we need to recall the following lemma proved in [17];

Lemma A.1. *Let $\zeta \in \Lambda^1 V^*$ and $\gamma \in \Lambda^r V^*$; we have*

$$\bigstar(\zeta \wedge \gamma) = (-1)^r \zeta \wedge \bigstar(\kappa \wedge \gamma) - (-1)^r \bigstar(\kappa \wedge \bigstar(\zeta \wedge \bigstar\gamma)). \tag{A.3}$$

Applying Eq. (A.3) with $\zeta = *(\nu_1 \wedge \kappa^2)$ and $\gamma = 1 \in \Lambda^0 M$ we have

$$(\bigstar\kappa) \wedge *(\nu_1 \wedge \kappa^2) = \bigstar(*(\nu_1 \wedge \kappa^2)) = *J(*(\nu_1 \wedge \kappa^2)) = -J\nu_1 \wedge \kappa^2. \tag{A.4}$$

Moreover, since $\nu_1 \in \Lambda^3_6 M$, it follows that

$$*(\nu_1 \wedge \kappa) \wedge \kappa = -J\nu_1 \wedge \kappa^2. \tag{A.5}$$

Eq. (A.4) together with Eq. (A.5) implies (A.2), so that Eq. (2.9) is proved. □

Proof of Theorem 3.4. In order to prove formula (3.11) it is useful to introduce the 1-forms $S_{ijk}\omega_k$, $V_{ik}\omega_k$, defined by the relations

$$\begin{aligned}
&\mathrm{d}T_{ij} = T_{ik}\theta_{kj} + T_{kj}\theta_{ki} + S_{ijk}\omega_k,\\
&\mathrm{d}M_i = M_k\theta_{ki} + V_{ik}\omega_k.
\end{aligned}$$

Using Eqs. (3.5) and (3.6) and the definition of T_{ij}, M_i given in (3.2)

$$\begin{aligned} D\tau_i &= \mathrm{d}T_{ij}\wedge\omega_j + T_{ij}\mathrm{d}\omega_j - 2\kappa_{ij}\mu\wedge\tau_j \\ &= (S_{iba} - T_{ij}T_{qa}\epsilon_{jbq} - T_{ij}\kappa_{jb}M_a - 2\kappa_{ij}M_aT_{jb})\omega_a\wedge\omega_b, \end{aligned}$$

and

$$\begin{aligned} D\mu &= \mathrm{d}M_r\wedge\omega_r + M_r\mathrm{d}\omega_r + \frac{2}{3}\kappa_{ij}\tau_i\wedge\tau_j \\ &= \left(V_{ba} - M_r\epsilon_{rbq}T_{qa} - M_r\kappa_{rb}M_a + \frac{2}{3}\kappa_{ij}T_{ia}T_{jb}\right)\omega_a\wedge\omega_b. \end{aligned}$$

Therefore, taking into account (3.8) and (3.9), we obtain

$$\begin{aligned} T_{iab} &= 2(S_{iba} - T_{ij}T_{qa}\epsilon_{jbq} - T_{ij}\kappa_{jb}M_a - 2\kappa_{ij}M_aT_{jb}), \\ N_{ab} &= 2\left(V_{ba} - M_r\epsilon_{rbq}T_{qa} - M_r\kappa_{rb}M_a + \frac{2}{3}\kappa_{ij}T_{ia}T_{jb}\right). \end{aligned}$$

It follows that

$$\begin{aligned} \epsilon_{ipq}T_{pqj} &= 2(\epsilon_{ipq}S_{pjq} - \epsilon_{ipq}\epsilon_{rjs}T_{pr}T_{sq} - \epsilon_{ipq}T_{pr}\kappa_{rj}M_q + 2\overline{\epsilon}_{iqr}T_{rj}M_q), \\ \kappa_{ip}N_{pj} &= 2\left(\kappa_{ip}V_{jp} - \kappa_{ip}\epsilon_{rjq}T_{qp}M_r - \kappa_{ip}\kappa_{rj}M_rM_p + \frac{2}{3}\kappa_{ip}\kappa_{qr}T_{qp}T_{rj}\right) \end{aligned}$$

and using the ϵ-identities (2.6)

$$\begin{aligned} \epsilon_{ipq}T_{pqi} &= 2(-\epsilon_{ipq}S_{ipq} - \epsilon_{ipq}\epsilon_{ris}T_{pr}T_{sq} - \overline{\epsilon}_{prq}T_{pr}M_q + 2\overline{\epsilon}_{qri}T_{ri}M_q) \\ &= 2(-\epsilon_{ipq}S_{ipq} - \epsilon_{ipq}\epsilon_{ris}T_{pr}T_{sq} + \overline{\epsilon}_{prq}T_{pr}M_q), \\ \kappa_{ip}N_{pi} &= 2\left(\kappa_{ip}V_{ip} - \kappa_{ip}\epsilon_{riq}T_{qp}M_r - \kappa_{ip}\kappa_{ri}M_rM_p + \frac{2}{3}\kappa_{ip}\kappa_{qr}T_{qp}T_{ri}\right) \\ &= 2\left(\kappa_{ip}V_{ip} + \overline{\epsilon}_{rqp}T_{qp}M_r + \frac{2}{3}\kappa_{ip}\kappa_{qr}T_{qp}T_{ri} + \sum_i M_i^2\right). \end{aligned}$$

Then by Theorem 3.2 we get

$$\begin{aligned} s &= 4(-\epsilon_{ipq}S_{ipq} - \epsilon_{ipq}\epsilon_{ris}T_{pr}T_{sq} + \overline{\epsilon}_{prq}T_{pr}M_q) - 6\left(\kappa_{ip}V_{ip} + \overline{\epsilon}_{rqp}T_{qp}M_r + \frac{2}{3}\kappa_{ip}\kappa_{qr}T_{qp}T_{ri} + \sum_i M_i^2\right) \\ &= -4\epsilon_{ipq}S_{ipq} - 4\epsilon_{ipq}\epsilon_{ris}T_{pr}T_{sq} - 2\overline{\epsilon}_{prq}T_{pr}M_q - 6\kappa_{ip}V_{ip} - 4\kappa_{ip}\kappa_{qr}T_{qp}T_{ri} - 6\sum_i M_i^2. \end{aligned}$$

Furthermore a straightforward computation gives the following formulae

$$\begin{aligned} \pi_0^2 &= \frac{4}{9}T_{ii}T_{jj}, \\ \sigma_0^2 &= \frac{4}{9}\kappa_{ij}\kappa_{sr}T_{ij}T_{sr}, \\ |\pi_2|^2 &= -\frac{4}{3}T_{ii}T_{jj} + 4T_{ij}^2 - 2\epsilon_{sra}\epsilon_{aij}T_{sr}T_{ij} + 4\kappa_{ir}\kappa_{js}T_{ij}T_{sr}, \\ |\sigma_2|^2 &= -2\epsilon_{sra}\epsilon_{aij}T_{sr}T_{ij} - \frac{4}{3}\kappa_{ij}\kappa_{ab}T_{ij}T_{ab} - 4T_{ij}T_{ji} + 4\sum_{ij}T_{ij}^2, \\ |\nu_1|^2 &= \epsilon_{ijk}\epsilon_{kab}T_{ij}T_{ab}, \\ |\nu_3|^2 &= 2T_{ij}^2 + 2T_{ij}T_{ji} - 2\kappa_{jr}\kappa_{is}T_{ij}T_{rs} - 2\kappa_{ir}\kappa_{js}T_{ij}T_{rs}, \\ \mathrm{d}^*\pi_1 &= -\epsilon_{sra}\epsilon_{aij}T_{sr}T_{ij} + 4\overline{\epsilon}_{ijk}T_{ij}M_k - \epsilon_{sra}S_{sra} - 3\kappa_{ij}V_{ij} - 3\sum_i M_i^2, \end{aligned}$$

$$d^*\nu_1 = -\epsilon_{sra}\epsilon_{aij}T_{sr}T_{ij} + \overline{\epsilon}_{ijk}T_{ij}M_k - \epsilon_{sra}S_{sra},$$
$$\langle \pi_1, \nu_1\rangle = \epsilon_{abk}\epsilon_{kij}T_{ab}T_{ij} - 3\overline{\epsilon}_{ijk}T_{ij}M_k.$$

Therefore we get

$$\begin{aligned}
&\frac{15}{2}\pi_0^2 + \frac{15}{2}\sigma_0^2 + 2d^*\pi_1 + 2d^*\nu_1 - |\nu_1|^2 - \frac{1}{2}|\sigma_2|^2 - \frac{1}{2}|\pi_2|^2 - \frac{1}{2}|\nu_3|^2 + 4\langle \pi_1, \nu_1\rangle \\
&\quad = 4T_{ii}T_{jj} + 4\kappa_{ij}\kappa_{sr}T_{ij}T_{sr} - 5\sum_{ij} T_{ij} + \epsilon_{sra}\epsilon_{aij}T_{sr}T_{ij} + T_{ij}T_{ji} - 2\overline{\epsilon}_{ijk}T_{ij}M_k \\
&\qquad - 6\kappa_{ij}V_{ij} - 6\sum_i M_i^2 + (-\kappa_{ia}\kappa_{jb} + \kappa_{ib}\kappa_{ja})T_{ij}T_{ba} - 4\epsilon_{ijk}S_{ijk} \\
&\quad = 4\epsilon_{ipq}S_{ipq} - 4\epsilon_{ipq}\epsilon_{ris}T_{pr}T_{sq} - 2\overline{\epsilon}_{prq}T_{pr}M_q - 6\kappa_{ip}V_{ip} - 4\kappa_{ip}\kappa_{qr}T_{qp}T_{ri} - 6\sum_i M_i^2,
\end{aligned}$$

i.e.

$$s = \frac{15}{2}\pi_0^2 + \frac{15}{2}\sigma_0^2 + 2d^*\pi_1 + 2d^*\nu_1 - |\nu_1|^2 - \frac{1}{2}|\sigma_2|^2 - \frac{1}{2}|\pi_2|^2 - \frac{1}{2}|\nu_3|^2 + 4\langle \pi_1, \nu_1\rangle,$$

and the theorem is proved. □

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